

On the Stability of a Class of Modified Gravitational Models

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Abstract Motivated by the dark energy issue, a minisuperspace approach to the stability for modified gravitational models in a four dimensional cosmological setting is investigated. Specifically, after revisiting the $f(R)$ case, R being the Ricci curvature, we present a stability condition around a de Sitter solution valid for modified gravitational models of generalized type $F(R, G, Q)$, G and Q being the Gauss-Bonnet and quadratic Riemann invariants respectively. A generalization to higher order invariants is presented.

1 Introduction

It is well known that recently it has been found strong evidence for an accelerated expansion of the universe, apparently due to the presence of an effective positive cosmological constant and associated with this acceleration there exists the so called dark energy issue (see for example [1]).

The modified gravity models are pure gravitational alternative for dark energy (for a recent review and alternative approaches see [2, 3]). The main idea underlying these approaches to dark energy puzzle is quite simple and consists in adding to the gravitational Einstein-Hilbert action other gravitational terms which may dominate the cosmological evolution during the very early or the very late universe epochs, but in such a way that General Relativity remains valid at intermediate epochs and also at non cosmological scale.

In the present paper we shall consider a large class of modified gravitational cosmological models defined in a Friedmann-Robertson-Walker (FRW) space-time and we shall focus our attention on the stability of the de Sitter solution. For the sake of simplicity we shall consider the phenomenological-important spatially flat case.

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We recall that the stability issue is relevant in many contexts. For example, in the Λ CDM model it ensures that no future singularity will be present in the solution and within cosmological models, the stability or instability around a de Sitter solution is of some interest at early or later times. However, we remind that the inclusion of a cosmological term has to confront with the well known cosmological constant problem, an unsolved issue so far. On the other hand, as anticipated before, the modified gravity models may offer a quite natural geometrical approach in the spirit of the original Einstein idea.

First we shall briefly revisit the class of models based on the action (here $\kappa^2 = 8\pi G_N$, G_N being the Newton's constant)

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R), \tag{1.1}$$

where $f(R)$ depends only on the scalar curvature R . We will consider pure gravitational theories, since we are interested in the dark energy sector, namely the property of the de Sitter critical point.

The method we shall present here is suitable for dealing with de Sitter critical points, that is the in absence of ordinary matter or radiation, but in principle, with simple modifications, it could be extended in order to include ordinary matter and this is important in reconstructing the expansion history of the Universe and probing the phenomenological relevance of the models (see for example the recent paper [4, 5], where the case $f(R)$ has been discussed).

In a cosmological setting, the $f(R)$ models have been introduced in [6–8] and investigated in many papers (see [3]). Also the stability of the solutions has been discussed in several places [4, 9–21]. To this aim, different techniques have been employed, including manifestly covariant and field theoretical approaches, where the gauge issue has been properly taken into account. All these investigations are in agreement with the following conditions which ensures the existence and the stability of the de Sitter solution:

$$2f_0 = R_0 f'_0, \tag{1.2}$$

$$1 < \frac{f'_0}{R_0 f''_0}, \tag{1.3}$$

where f' , f'' are the derivatives of $f(R)$ with respect to R and f'_0 , f''_0 are the derivatives evaluated at the value $R = R_0$. The first condition, (1.2), determines the scalar curvature of the de Sitter solution, while the second one, (1.3), gives the condition for the stability around the de Sitter solution. In a more general quadratic theory, the stability has been investigated in [22, 23].

The aim of this paper is to address the same investigation to a more general class of modified gravity, that is the one based on the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} F(R, G, Q), \tag{1.4}$$

which depend on an arbitrary function of the scalar curvature R , Gauss-Bonnet invariant G and the quadratic Riemann invariant Q , the relation among them being

$$G = R^2 - 4P + Q, \quad P = R_{\mu\nu} R^{\mu\nu}, \quad Q = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \tag{1.5}$$

The string-inspired scalar-Gauss-Bonnet gravity case $F(R, G)$ has been suggested in [24] as a model for gravitational dark energy while some time ago it has been applied to the possible solution of the initial singularity problem [25–27].

The investigation of different regimes of cosmic acceleration in such string-inspired gravity models has been carried out in [24, 28–43]. In particular, in [41] a first attempt to the study of the stability of such kind of models has been carried out using an approach based on quantum field theory.

The method we will use in order to study the stability for $F(R, G, Q)$ models is a classical approach, which we call minisuperspace approach, and it is based on a Lagrangian formalism [44–46], inspired by the seminal paper [47], where, for the first time, quantum effects were considered.

With regard to this, it is well known that one-loop and two-loop quantum effects induce higher derivative gravitational terms in the effective gravitational Lagrangian and early studies on instability for quadratic terms have been investigated in [48–50]. A particular case has been recently studied in [51].

For general quadratic models, namely with linear dependence on R^2 , P and Q , the stability has been reported in [22, 23].

A stability analysis of nontrivial vacua in a general class of higher-derivative theories of gravitation has already been presented in [57]. Our approach is different from the one presented there in the sense we are dealing with scalar quantities instead of tensorial ones and, as a result, our general stability condition of Sect. 4 is exhibited in a more explicit and useful form.

It should be stressed that the stability studied here is the one with respect to homogeneous perturbations. For the $F(R)$ case, the stability criterion for homogeneous perturbations coincides with the inhomogeneous ones [10–12].

The content of the paper is the following. In Sect. 2, we revisit the $f(R)$ models and rederive the stability condition (1.3) and then the same analysis is extended to the Gauss-Bonnet models in Sect. 3 and to the general case in Sect. 4. The paper ends with the Sect. 5.

2 Minisuperspace Approach for $f(R)$

As already mentioned above, we shall deal with FRW isotropic and homogeneous models with spatial flat metric, namely

$$ds^2 = -N^2(t)dt^2 + a^2(t)d^2\vec{x}, \tag{2.1}$$

where t is the cosmic time, $a(t)$ the cosmological factor and $N(t)$ an arbitrary lapse function, which describes the gauge freedom associated with the reparametrization invariance of the minisuperspace gravitational model. For the above metric, the scalar curvature reads

$$R = 6 \left(\frac{\ddot{a}}{aN^2} + \frac{\dot{a}^2}{a^2N^2} - \frac{\dot{a}\dot{N}}{aN^3} \right). \tag{2.2}$$

As usual the ‘dot’ over the symbol means derivative with respect to time, so $\dot{a} = \frac{da}{dt}$ and so on.

If one plugs this expression in the (1.1), one obtains a higher derivative Lagrangian theory. In order to work with a standard (first derivatives) Lagrangian system, we make use of a Lagrangian multiplier λ and we write [44, 45]

$$S = \frac{1}{2\kappa^2} \int d\vec{x} \int dt Na^3 \left[f(R) - \lambda \left[R - 6 \left(\frac{\ddot{a}}{aN^2} + \frac{\dot{a}^2}{a^2N^2} - \frac{\dot{a}\dot{N}}{aN^3} \right) \right] \right]. \tag{2.3}$$

Making the variation with respect to R , one gets

$$\lambda = \frac{df(R)}{dR}. \tag{2.4}$$

Thus, substituting this value and making an integration by part one arrives at the Lagrangian, which will be our starting point

$$L(a, \dot{a}, R, \dot{R}, N) = -6 \frac{\dot{a}^2 a}{N} \frac{df(R)}{dR} - 6 \frac{\dot{a} a^2 \dot{R}}{N} \frac{d^2 f(R)}{d^2 R} + Na^3 \left[f(R) - R \frac{df(R)}{dR} \right]. \tag{2.5}$$

Now, a and R are Lagrangian variables and N appears as an “einbein” Lagrangian multiplier, reflecting the parametrization invariance of the action. Furthermore, the three equations of motion related to these three variables are not independent as we will show. In the analysis of the system we can use only two equations of motion and we can fix $N(t)$ by the gauge choice (this corresponds to the choice of the cosmological time).

The first Equation of motion $\frac{\partial L}{\partial N} = 0$ reads

$$\dot{f}' H + f' H^2 + \frac{1}{6}(f - Rf') = 0, \tag{2.6}$$

where for convenience we have introduced

$$H = \frac{\dot{a}}{a}, \quad f' = \frac{df}{dR}, \tag{2.7}$$

and we have chosen the gauge $N(t) = 1$.

The conserved quantity related to the parametrization invariance is the energy computed with the standard Legendre transformation and it is vanishing on shell due to the equation of motion (2.6) for the einbein N . This is the energy constraint.

The other equations of motion associated with the variation of R and a are respectively

$$\dot{H} + 2H^2 - \frac{R}{6} = 0, \tag{2.8}$$

$$2\ddot{f}' + 4\dot{f}' H - 2H^2 f' - \frac{Rf'}{3} + f = 0. \tag{2.9}$$

As we already said above, this latter equation is redundant. In fact, taking the derivative with respect to t of (2.6) and making use of (2.8) a direct calculation leads to (2.9), while (2.8) is equivalent to (2.2).

Equations (2.6) and (2.8) form a very simple autonomous system in the two variables R and H , namely

$$\dot{R} = -\frac{1}{f''} \left(f' H + \frac{f - Rf'}{6H} \right), \tag{2.10}$$

$$\dot{H} = \frac{R}{6} - 2H^2. \tag{2.11}$$

The analysis of stability is standard and consists first in finding the critical points R_0, H_0 , which are defined by imposing $\dot{R} = 0$ and $\dot{H} = 0$ and then in investigating the stability of

the related linear system around such critical points. In our case we have the solutions

$$R_0 = 12H_0^2, \tag{2.12}$$

$$2f_0 - R_0 f'_0 = 0, \tag{2.13}$$

which correspond to a de Sitter critical point with scalar curvature determined by the above condition.

The linearized system around the de Sitter critical point reads

$$\begin{pmatrix} \delta \dot{R} \\ \delta \dot{H} \end{pmatrix} = M \begin{pmatrix} \delta R \\ \delta H \end{pmatrix}, \quad M = \begin{pmatrix} H_0 & -\frac{4f_0}{R_0 f'_0} \\ \frac{1}{6} & -4H_0 \end{pmatrix}, \tag{2.14}$$

where M is the stability matrix. It is easy to show that the stability is ensured by the two conditions $\text{Tr } M < 0$ and $\det M > 0$. The first one is trivially satisfied, while the second one, making use of (2.13) leads to the stability condition (1.3).

3 Minisuperspace Approach for the Gauss-Bonnet Model

In this section we shall generalize the previous approach to the modified gravitational model defined by a Lagrangian density of the type $F(R, G)$. Here G is the Gauss-Bonnet invariant, which for the FWR metric reads

$$G = \frac{24\dot{a}^2}{a^3 N^5} (\ddot{a}N - \dot{a}\dot{N}), \tag{3.1}$$

while R is given by (2.2).

In order to put the Lagrangian in a standard form, now we have to use two Lagrangian multipliers, so we start from

$$\begin{aligned} S = & \frac{1}{2\kappa^2} \int d\vec{x} \int dt Na^3 \left\{ F(R, G) - \lambda \left[R - 6 \left(\frac{\ddot{a}}{aN^2} + \frac{\dot{a}^2}{a^2 N^2} - \frac{\dot{a}\dot{N}}{aN^3} \right) \right] \right. \\ & \left. - \mu \left[G - \frac{24\dot{a}^2}{a^3 N^5} (\ddot{a}N - \dot{a}\dot{N}) \right] \right\}. \end{aligned} \tag{3.2}$$

Making the variation with respect to R and G one gets

$$\lambda = \frac{\partial F(R, G)}{\partial R} \equiv F'_R, \quad \mu = \frac{\partial F(R, G)}{\partial G} \equiv F'_G, \tag{3.3}$$

and by making an integration by part one arrives at the Lagrangian

$$\begin{aligned} L(a, \dot{a}, R, \dot{R}, G, \dot{G}, N) = & -\frac{6\dot{a}^2 a F'_R}{N} - \frac{8\dot{a}^3}{N^3} \frac{dF'_G}{dt} - \frac{6\dot{a}a^2}{N} \frac{dF'_R}{dt} \\ & + Na^3 (F - RF'_R - GF'_G). \end{aligned} \tag{3.4}$$

Of course, the couple of equations of motion related to the variations of R and G are equivalent to (2.2) and (3.1), which lead to

$$R = 6(\dot{H} + 2H^2), \quad G = 4H^2(R - 6H^2), \tag{3.5}$$

while the equation $\frac{\partial L}{\partial N} = 0$ gives

$$24H^3 \dot{F}'_G + 6H^2 F'_R + 6H \dot{F}'_G + (F - RF'_R - GF'_G) = 0, \tag{3.6}$$

where we have chosen $N(t) = 1$ again and we have put $H(t) = \dot{a}(t)/a(t)$ as above. The conserved quantity is always the energy computed with the standard Legendre transformation and it is vanishing on shell.

The equation of motion related to the variation of a is

$$8H^2 \ddot{F}'_G + 2\ddot{F}'_R + 4H \dot{F}'_R + 16H \dot{F}'_G (\dot{H} + H^2) + F'_R (4\dot{H} + 6H^2) + F - RF'_R - GF'_G = 0. \tag{3.7}$$

Again, this latter equation is a consequence of the others. In fact, taking the derivative with respect to t of (3.6) and making use of (3.5), one obtains (3.7). As a consequence, in order to deal with a first order differential autonomous system, we may use (3.5) and (3.6), obtaining in this way

$$\dot{R} = \frac{B(R, H)}{A(R, H)}, \tag{3.8}$$

$$\dot{H} = \frac{R}{6} - 2H^2, \tag{3.9}$$

where

$$A(R, H) = 6HF'_R + 48H^3 F''_{RG} + 96H^5 F''_{GG}, \tag{3.10}$$

$$B(R, H) = -F + (R - 6H^2)(F'_R + 4H^2 F'_G) - 8H^2(R - 12H^2)^2(F''_{RG} + 4H^2 F''_{GG}). \tag{3.11}$$

The critical points are defined by $\dot{R} = 0$ and $\dot{H} = 0$. As a result

$$R_0 = 12H_0^2, \quad G_0 = 24H_0^4, \tag{3.12}$$

and from $B(R_0, H_0) = 0$ it follows

$$F_0 - G_0 F'_G(R_0, G_0) - \frac{R_0 F'_R(R_0, G_0)}{2} = 0. \tag{3.13}$$

This corresponds to a de Sitter critical point with Gauss-Bonnet invariant determined by the condition (3.13) (see [41]).

Since

$$\frac{1}{A_0} \frac{\partial B_0}{\partial R} = H_0, \tag{3.14}$$

the linearized system around de Sitter critical point reads

$$\begin{pmatrix} \delta \dot{R} \\ \delta \dot{H} \end{pmatrix} = M \begin{pmatrix} \delta R \\ \delta H \end{pmatrix}, \quad M = \begin{pmatrix} H_0 & -\frac{12F'_R(R_0, G_0)}{A_0} \\ \frac{1}{6} & -4H_0 \end{pmatrix}, \tag{3.15}$$

where M is the stability matrix and A_0, B_0 are quantities in (3.10) and (3.11) evaluated at R_0 and H_0 . Again, requiring $\text{Tr } M < 0$ (again trivially satisfied) and $\det M > 0$ one obtains

the stability condition

$$1 < \frac{F'_R(R_0, G_0)}{R_0[F''_{RR}(R_0, G_0) + \frac{2}{3}R_0F''_{GR}(R_0, G_0) + \frac{1}{9}R_0^2F''_{GG}(R_0, G_0)]}. \tag{3.16}$$

It is easy to show that this condition, when $F(R, G) = f(R)$ reproduces the relation (1.3) discussed in Sect. 2.

It has to be noted that the same result can be obtained starting from (3.7) and reducing it to a first order autonomous system by the method discussed in [4], but in this case, the resulting stability matrix is a 3×3 matrix.

We conclude the section with an example. We recall that it is interesting to investigate Gauss-Bonnet models defined by

$$F(R, G) = R + f(G), \tag{3.17}$$

since they may be relevant from a phenomenological point of view [41, 42]. In these special cases the conditions (3.13) and (3.16) read

$$G_0f'_0 - f_0 = 6H_0^2, \tag{3.18}$$

$$1 < \frac{9}{R_0^3f''_0}. \tag{3.19}$$

As a particular example let us choose

$$f(G) = \alpha G^\gamma, \tag{3.20}$$

where α is a dimensional constant and γ is a dimensionless parameter. Equation (3.18) gives

$$2\alpha(\gamma - 1)G_0^{\gamma-1/2} = \sqrt{6} \tag{3.21}$$

and this implies that $\alpha(\gamma - 1) > 0$, while (3.19) leads to $1 < 1/(2\gamma)$. Then the model is stable (around de Sitter) if the arbitrary parameters α and β satisfy both the conditions

$$\begin{cases} \alpha(\gamma - 1) > 0 \\ \frac{1}{2\gamma} > 1 \end{cases} \implies \begin{cases} \alpha < 0 \\ 0 < \gamma < \frac{1}{2}. \end{cases} \tag{3.22}$$

All the other possible choices for the parameters α, β give rise to an unstable de Sitter solution. In particular, the model with negative parameters α and γ is unstable.

4 Minisuperspace Approach for the $F(R, G, Q)$ Case

Now we shall generalize the approach described in the previous section to a Lagrangian density of the type $F(R, G, Q)$. The use of G instead of P simplifies the derivation. Here Q is the quadratic Riemann invariant, which for the FWR metric (2.1) reads

$$Q = \frac{12}{a^2N^6}(\ddot{a}N - \dot{a}\dot{N})^2 + \frac{12\dot{a}^4}{a^4N^4}. \tag{4.1}$$

In order to put the Lagrangian in a standard form, now we have to use three Lagrangian multipliers, so we start from

$$S = \frac{1}{2\kappa^2} \int d\bar{x} \int dt Na^3 \left\{ F(R, G, Q) - \lambda \left[R - 6 \left(\frac{\ddot{a}}{aN^2} + \frac{\dot{a}^2}{a^2N^2} - \frac{\dot{a}\dot{N}}{aN^3} \right) \right] - \mu \left[G - \frac{24\dot{a}^2}{a^3N^5} (\ddot{a}N - \dot{a}\dot{N}) \right] - \rho \left[Q - \left(\frac{12}{a^2N^6} (\ddot{a}N - \dot{a}\dot{N})^2 + \frac{12\dot{a}^4}{a^4N^4} \right) \right] \right\}. \tag{4.2}$$

Making the variation with respect to R, G and Q , one gets

$$\lambda = \frac{\partial F}{\partial R} \equiv F'_R, \quad \mu = \frac{\partial F}{\partial G} \equiv F'_G, \quad \rho = \frac{\partial F}{\partial Q} \equiv F'_Q, \tag{4.3}$$

and by making an integration by part, one arrives at the Lagrangian

$$L = -\frac{6\dot{a}^2 a F'_R}{N} - \frac{8\dot{a}^3}{N^3} \frac{dF'_G}{dt} - \frac{6\dot{a}a^2}{N} \frac{dF'_R}{dt} + Na^3 (F - RF'_R - GF'_G - QF'_Q) + 12F'_Q \left(\frac{\dot{a}^4}{aN^3} + \frac{a\ddot{a}^2}{N^3} + \frac{a\dot{a}^2\dot{N}^2}{N^5} - \frac{2a\dot{a}\ddot{a}\dot{N}}{N^4} \right). \tag{4.4}$$

The equations of motion related to the variations of R, G and Q are equivalent to (2.2), (3.1) and (4.1) respectively and read

$$R = 6(\dot{H} + 2H^2), \quad G = 4H^2(R - 6H^2), \quad Q = 12H^4 + \frac{1}{3}(R - 6H^2)^2. \tag{4.5}$$

Here we have put $H(t) = \dot{a}(t)/a(t)$ and we have choose the “gauge” $N(t) = 1$. The equation of motion for the variable N gives (in the gauge $N(t) = 1$)

$$24H^3 \dot{F}'_G + 6H^2 F'_R + 6H \dot{F}'_R + (F - RF'_R - GF'_G - QF'_Q) + 24H(\dot{H} + H^2)\dot{F}'_Q + 12F'_Q(2H\ddot{H} + 6\dot{H}H^2 - \dot{H}^2) = 0, \tag{4.6}$$

while the equation of motion related to the variation of a again is a consequence of the other ones and for this reason we do not write it.

Now making use (4.5) and (4.6), we arrive at the following first order differential autonomous system again composed only by two equations,

$$\dot{R} = \frac{B(R, H)}{A(R, H)}, \tag{4.7}$$

$$\dot{H} = \frac{R}{6} - 2H^2, \tag{4.8}$$

where

$$A(R, H) = 4HF'_Q + 6H[F''_{RR} + 8H^3(F''_{RG} + F''_{RQ})] + [16H^4(F''_{GG} + 2F''_{GQ} + F''_{QQ})] + A_1, \tag{4.9}$$

$$B(R, H) = -F + RF'_R + GF'_G + QF'_Q - 6H^2 F'_R + 4H^2 F'_Q (R - 12H^2) + B_1, \tag{4.10}$$

where A_1 and B_1 are trivially vanishing when evaluated at the critical points $\dot{R} = 0$ and $\dot{H} = 0$. They are given by

$$A_1 = 8H(R - 12H^2) \left(F''_{RQ} + 4H^2 F''_{GQ} + \frac{R}{3} F''_{QQ} \right), \tag{4.11}$$

$$B_1 = -8H(R - 12H^2)^2 (F''_{RG} + 4H^2 F''_{GG} - F''_{QR}). \tag{4.12}$$

As a result, one has

$$R_0 = 12H_0^2, \quad G_0 = Q_0 = 24H_0^4 \tag{4.13}$$

and from $B(R_0, H_0) = 0$, it follows

$$F_0 - \frac{R_0^2}{6} (F'_G + Q_0 F'_Q) - \frac{R_0 F'_R}{2} = 0. \tag{4.14}$$

This generalizes the condition (3.13).

The linearized system around de Sitter critical point reads

$$\begin{pmatrix} \delta \dot{R} \\ \delta \dot{H} \end{pmatrix} = M \begin{pmatrix} \delta R \\ \delta H \end{pmatrix}, \quad M = \begin{pmatrix} H_0 & -\frac{12H_0(F'_R + 8H_0^2 F'_Q)}{A_0} \\ \frac{1}{6} & -4H_0 \end{pmatrix}, \tag{4.15}$$

$A_0 = A(R_0, H)$ being the quantity in (4.9) evaluated at the critical point. Again the trace of the 2×2 matrix M is equal to the $-3H_0 < 0$, while from $\det M > 0$, we get the stability condition

$$1 < \frac{F'_R + \frac{2}{3} R_0 F'_Q}{R_0 [F''_{RR} + \frac{2}{3} F'_Q + \frac{2}{3} R_0 (F''_{GR} + F''_{RQ}) + \frac{1}{9} R_0^2 (F''_{GG} + 2F''_{GQ} + F''_{QQ})]}. \tag{4.16}$$

This is the main result of our paper. If F does not depend on Q , the above condition reduces (3.16) derived in Sect. 3.

Let us consider an example, namely the model defined by [52]

$$F(R, G, Q) = R - \mu^2 f(R, G, Q), \tag{4.17}$$

$$f(R, G) = \frac{1}{(\alpha R^2 + 6\beta G + 6\gamma Q)^n}, \quad f_0 = f(R_0, G_0, Q_0) = \frac{1}{R_0^{2n} (\alpha + \beta + \gamma)^n}, \tag{4.18}$$

where $\mu^2, \alpha, \beta, \gamma$ are arbitrary dimensional and dimensionless parameters and n is an arbitrary integer number. Now from (3.13) and (3.16) we respectively obtain

$$\frac{R_0}{2} - (n + 1)\mu^2 f_0 = 0 \implies f_0 > 0 \implies (\alpha + \beta + \gamma)^n > 0, \tag{4.19}$$

$$\frac{\alpha + \beta + \gamma}{\omega_n(\alpha, \beta, \gamma)} < 0, \quad \omega_n(\alpha, \beta, \gamma) = 2n(\alpha + \beta + \gamma) + 2\beta + \alpha. \tag{4.20}$$

Equation (4.19) ensures the existence of the de Sitter solution while (4.20) is necessary for the stability of such a solution.

We immediately see that (4.20) is not satisfied if all the parameters α, β and γ have the same sign, but it is not satisfied also in some other cases. For example, it is never satisfied if both $\alpha + \beta + \gamma$ and $\omega_n(\alpha, \beta, \gamma)$ have the same sign. The cases in which $\alpha + \beta + \gamma$ and $\omega_n(\alpha, \beta, \gamma)$ have opposite signs require a more detailed analysis.

As an example, we may take $\alpha = -\gamma, \gamma > 0, \beta > 0$ thus $\alpha + \beta + \gamma = \beta > 0$. Then, as soon as $\beta < \frac{\gamma}{2(n+1)}$, one has a De Sitter stable solution.

We conclude this section observing that the method we are dealing with is very suitable for generalization to the case where higher curvature invariants of the curvature are present. An interesting example is the third order invariant Q_3 , given by

$$Q_3 = R_{\mu\nu\alpha\beta} R^{\alpha\beta\rho\sigma} R^{\mu\nu} = 24 \frac{\dot{N}^3 \dot{a}^3}{a^3 N^9} - 72 \frac{\dot{N}^2 \dot{a}^2 \ddot{a}}{a^3 N^8} + 72 \frac{\dot{N} \dot{a} \ddot{a}^2}{a^3 N^7} - 24 \frac{\ddot{a}^3}{a^3 N^6} - 24 \frac{\dot{a}^6}{a^6 N^6}. \tag{4.21}$$

We recall that this is the invariant which governs the two-loop ultraviolet divergences in pure gravity [53, 54]. In this case, one has

$$F_0 - \frac{R_0^2}{6} (F'_G + F'_Q) - \frac{R_0 F'_R}{2} - \frac{3}{2} Q_3 F'_Q = 0, \tag{4.22}$$

$$1 < \frac{F'_R + \frac{2}{3} R_0 F'_Q + 36 H_0^4 F'_Q}{R_0 [F''_{RR} + \frac{2}{3} F''_{Q} + \frac{2}{3} R_0 (F''_{GR} + F''_{RQ}) + \frac{1}{9} R_0^2 (F''_{GG} + 2F''_{GQ} + F''_{QQ})] + K_3}, \tag{4.23}$$

with

$$K_3 = 48 H_0^4 F'_{Q_3} + 288 H_0^6 (F''_{RQ_3} + 4 H_0^2 (F''_{GQ_3} + F''_{Q_3}) + 6 H_0^4 F''_{Q_3}). \tag{4.24}$$

If now we consider the special case $F(R, G, Q, Q_3) = f(R, G, Q) - bQ_3$, then a straightforward calculation leads to the following relations for the existence and the stability of the de Sitter solution:

$$F_0 - \frac{R_0^2}{6} (F'_G + F'_Q) - \frac{R_0 F'_R}{2} - \frac{b}{24} R_0^3 = 0, \tag{4.25}$$

$$1 < \frac{F'_R + \frac{2}{3} R_0 F'_Q + \frac{b}{4} R_0^2}{R_0 [F''_{RR} + \frac{2}{3} F''_{Q} + \frac{2}{3} R_0 (F''_{GR} + F''_{RQ}) + \frac{1}{9} R_0^2 (F''_{GG} + 2F''_{GQ} + F''_{QQ})] + \frac{b}{3} R_0^2}. \tag{4.26}$$

As an explicit example of this type, we consider the function

$$F = R - 2\Lambda_0 + a_1 R^2 + a_2 P + a_3 Q - bQ_3, \tag{4.27}$$

where P is given by (1.5) and in order to use (4.22) it has to be replaced in terms of R, G and Q . From (4.22), we get

$$R_0 - 4\Lambda_0 - \frac{b}{36} R_0^3 = 0, \tag{4.28}$$

while condition (4.23) leads to

$$\frac{12 - bR_0^2}{(3a_1 + a_2 + a_3)R_0 + \frac{b}{2}R_0^2} > 0. \tag{4.29}$$

For Λ small, one has real positive solutions. When $b = 0$ (the absence of the cubic term), the stability condition is

$$a_2 + a_3 + 3a_1 > 0, \tag{4.30}$$

which is the result obtained in [23]. In particular, when $a_2 = 0$ and $a_3 = 0$, one obtains the well known result that models of the type $R - 2\Lambda_0 + a_1 R^2$ are stable as soon as $a_1 > 0$.

As a final remark, one should note that for $b > 0$, besides the flat solution one can also have a de Sitter solution induced by quantum effects [47] starting with $\Lambda_0 = 0$ in the Lagrangian, the scalar curvature being $R_0^2 = 36/b$. As it is well known, and evident from (4.28), this is not true for the quadratic case, which requires a non vanishing cosmological constant term Λ_0 in the initial Lagrangian. In this case, the stability condition is satisfied when

$$(a_2 + a_3 + 3a_1) < -3\sqrt{b}. \tag{4.31}$$

5 Conclusions

In this paper we have presented the Lagrangian minisuperspace approach to the stability issue around a de Sitter critical point for a class of modified gravitational models depending on the Ricci scalar, Gauss-Bonnet and Quadratic Riemann invariants. It should be stressed that the stability studied here is the one with respect to homogeneous perturbations. For the $F(R)$ case, the stability criterion for homogeneous perturbations coincides with the inhomogeneous ones [10–12].

With respect to other approaches, the method here presented has the advantage that the autonomous system one is dealing with, as far as the stability of de Sitter solution is concerned, consists *always* in two first order differential equations, simplifying considerably the stability analysis for *arbitrary* dependence on the chosen curvature invariants. This is true only for spatially flat space-times.

The method has been first applied to the class of models depending only on the Ricci scalar R and the well known stability condition for this case has been recovered. Then the same approach has been applied to the scalar-Gauss-Bonnet models and its generalizations and a new general condition for the stability around a de Sitter solution has been found.

With regard to this last issue, we would like to note that the conditions (4.22) (existence of the de Sitter solution) and (4.23) (stability condition around the de Sitter solution) are very general and they can be applied to a large class of Lagrangians, as shown in the last example of Sect. 4, where a third order invariant in the Riemann tensor has been investigated.

As far as the comparison with other methods is concerned, in reference [51], a criterion of stability for a class of generalized modified gravitational models around de Sitter solution and described by a function of the kind $W(R, Q - 4P)$ has been derived with a field theoretical approach. Since $G = R^2 - 4P + Q$, our stability condition can easily be related to the one obtained within a theoretical approach.

Furthermore, we also have to mention [55], where the model of [52] have been discussed. The model considered in that paper is described by the choice

$$F(R, P, Q) = R - \frac{\mu^2}{(aR^2 + bP + cQ)^n}, \tag{5.1}$$

with a, b, c positive parameters and n a positive integer. The instability of the model is due to the appearance of a spin-2 ghost. As a result, an instability of the vacuum arises. In our notation, this model corresponds to the first model (4.17) we have studied in Sect. 4 and we have

$$\alpha = a + \frac{b}{4}, \quad \beta = -\frac{b}{4}, \quad \gamma = c + \frac{b}{4}. \tag{5.2}$$

When a , b , c and n are positive, our condition (4.16) is not satisfied and, as a consequence, the model is unstable, in agreement with the conclusions reported in [55]. Thus, the instability is a generic feature of this class of modified models. Within a theoretical approach, this has been already noted in [56].

Finally, we observe that the inclusion of the matter for a generic F might be non trivial, but, within our approach, can be done, for example, along the line of [4, 5, 58, 59].

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